10.2 Plane Curves and Parametric Equations

- **Sketch the graph of a curve given by a set of parametric equations.**
- **Eliminate the parameter in a set of parametric equations.**
- **Find a set of parametric equations to represent a curve.**
- **Understand two classic calculus problems, the tautochrone and brachistochrone problems.**

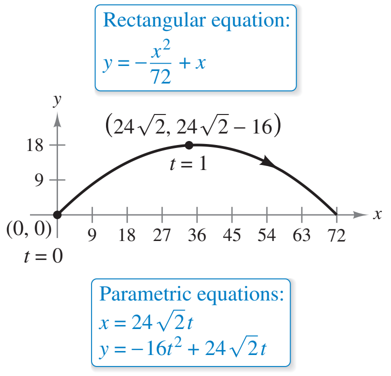
Plane Curves and Parametric Equations

Until now, you have been representing a graph by a single equation involving *two* variables. In this section, you will study situations in which *three* variables are used to represent a curve in the plane.

Consider the path followed by an object that is propelled into the air at an angle of 45°. For an initial velocity of 48 feet per second, the object travels the parabolic path given by

$$y = -\frac{x^2}{72} + x \qquad \text{Rectangular equation}$$

as shown in Figure 10.19. This equation, however, does not tell the whole story. Although it does tell you *where* the object has been, it doesn't tell you *when* the object was at a given point (x, y). To determine this time, you can introduce a third variable t, called a **parameter.** By writing both x and y as functions of t, you obtain the **parametric equations**



Curvilinear motion: two variables for position, one variable for time

Figure 10.19

$$x = 24\sqrt{2}t \qquad \text{Parametric equation for } x$$

and

$$y = -16t^2 + 24\sqrt{2}t. \qquad \text{Parametric equation for } y$$

From this set of equations, you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point

$$\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$$

and so on. (You will learn a method for determining this particular set of parametric equations—the equations of motion—later, in Section 12.3.)

For this particular motion problem, x and y are continuous functions of t, and the resulting path is called a **plane curve.**

REMARK At times, it is important to distinguish between a graph (the set of points) and a curve (the points together with their defining parametric equations). When it is important, the distinction will be explicit. When it is not important, C will be used to represent either the graph or the curve.

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are **parametric equations** and t is the **parameter.** The set of points (x, y) obtained as t varies over the interval I is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve,** denoted by C.

When sketching a curve represented by a set of parametric equations, you can plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in order of increasing values of t, the curve is traced out in a specific direction. This is called the **orientation** of the curve.

EXAMPLE 1 Sketching a Curve

Sketch the curve described by the parametric equations

$$x = f(t) = t^2 - 4$$

and

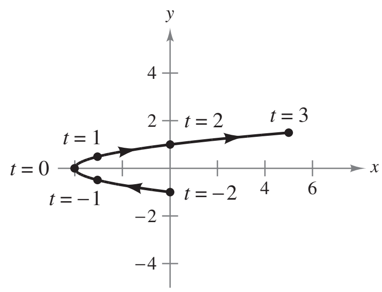
$$y = g(t) = \frac{t}{2}$$

where $-2 \le t \le 3$.

Solution For values of t on the given interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

By plotting these points in order of increasing t and using the continuity of f and g, you obtain the curve C shown in Figure 10.20. Note that the arrows on the curve indicate its orientation as t increases from -2 to 3.



Parametric equations:
$x = t^2 - 4$ and $y = \frac{t}{2}, -2 \le t \le 3$

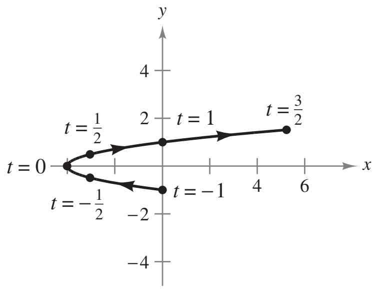
Figure 10.20

According to the Vertical Line Test, the graph shown in Figure 10.20 does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.

It often happens that two different sets of parametric equations have the same graph. For instance, the set of parametric equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \quad -1 \le t \le \frac{3}{2}$$

has the same graph as the set given in Example 1. (See Figure 10.21.) However, comparing the values of t in Figures 10.20 and 10.21, you can see that the second graph is traced out more *rapidly* (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.



Parametric equations:
$x = 4t^2 - 4$ and $y = t, -1 \le t \le \frac{3}{2}$

Figure 10.21

TECHNOLOGY Most graphing utilities have a *parametric* graphing mode. If you have access to such a utility, use it to confirm the graphs shown in Figures 10.20 and 10.21. Does the curve given by the parametric equations

$$x = 4t^2 - 8t \quad \text{and} \quad y = 1 - t, \quad -\frac{1}{2} \le t \le 2$$

represent the same graph as that shown in Figures 10.20 and 10.21? What do you notice about the *orientation* of this curve?

Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter.** For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.

Parametric equations	→	Solve for t in one equation.	→	Substitute into second equation.	→	Rectangular equation
$x = t^2 - 4$ $y = t/2$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

Once you have eliminated the parameter, you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$, as shown in Figure 10.20.

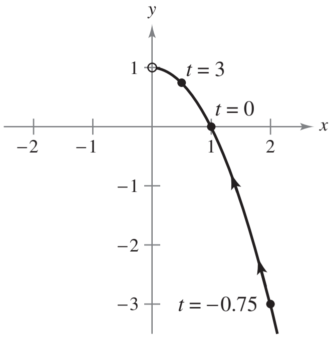
The range of x and y implied by the parametric equations may be altered by the change to rectangular form. In such instances, the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations. Such a situation is demonstrated in the next example.

EXAMPLE 2 Adjusting the Domain

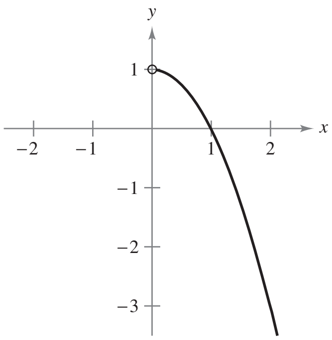
Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.



Parametric equations:
$x = \dfrac{1}{\sqrt{t+1}}, y = \dfrac{t}{t+1}, t > -1$



Rectangular equation:
$y = 1 - x^2, x > 0$

Figure 10.22

Solution Begin by solving one of the parametric equations for t. For instance, you can solve the first equation for t as follows.

$$x = \frac{1}{\sqrt{t+1}} \qquad \text{Parametric equation for } x$$

$$x^2 = \frac{1}{t+1} \qquad \text{Square each side.}$$

$$t + 1 = \frac{1}{x^2}$$

$$t = \frac{1}{x^2} - 1$$

$$t = \frac{1 - x^2}{x^2} \qquad \text{Solve for } t.$$

Now, substituting into the parametric equation for y produces

$$y = \frac{t}{t+1} \qquad \text{Parametric equation for } y$$

$$y = \frac{(1 - x^2)/x^2}{[(1 - x^2)/x^2] + 1} \qquad \text{Substitute } (1 - x^2)/x^2 \text{ for } t.$$

$$y = 1 - x^2. \qquad \text{Simplify.}$$

The rectangular equation, $y = 1 - x^2$, is defined for all values of x, but from the parametric equation for x, you can see that the curve is defined only when $t > -1$. This implies that you should restrict the domain of x to positive values, as shown in Figure 10.22.

It is not necessary for the parameter in a set of parametric equations to represent time. The next example uses an *angle* as the parameter.

EXAMPLE 3 Using Trigonometry to Eliminate a Parameter

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the curve represented by

$$x = 3 \cos \theta \quad \text{and} \quad y = 4 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.

Solution Begin by solving for $\cos \theta$ and $\sin \theta$ in the given equations.

$$\cos \theta = \frac{x}{3} \qquad \text{Solve for } \cos \theta.$$

and

$$\sin \theta = \frac{y}{4} \qquad \text{Solve for } \sin \theta.$$

Next, make use of the identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

to form an equation involving only x and y.

$$\cos^2 \theta + \sin^2 \theta = 1 \qquad \text{Trigonometric identity}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \text{Substitute.}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{Rectangular equation}$$

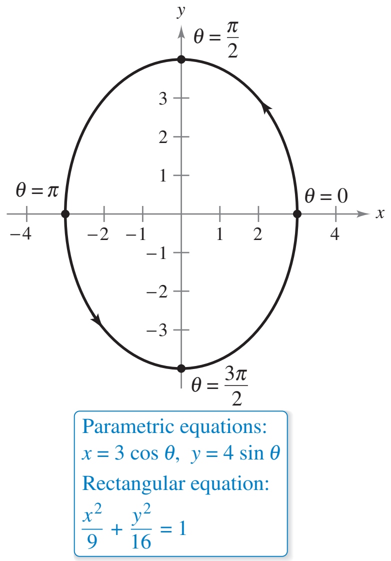


Figure 10.23

From this rectangular equation, you can see that the graph is an ellipse centered at $(0, 0)$, with vertices at $(0, 4)$ and $(0, -4)$ and minor axis of length $2b = 6$, as shown in Figure 10.23. Note that the ellipse is traced out *counterclockwise* as θ varies from 0 to 2π.

Using the technique shown in Example 3, you can conclude that the graph of the parametric equations

$$x = h + a \cos \theta \quad \text{and} \quad y = k + b \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

is the ellipse (traced counterclockwise) given by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

The graph of the parametric equations

$$x = h + a \sin \theta \quad \text{and} \quad y = k + b \cos \theta, \quad 0 \leq \theta \leq 2\pi$$

is also the ellipse (traced clockwise) given by

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

TECHNOLOGY Use a graphing utility in *parametric* mode to graph several ellipses.

In Examples 2 and 3, it is important to realize that eliminating the parameter is primarily an *aid to curve sketching.* When the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to tell you the *position, direction,* and *speed* at a given time.

Finding Parametric Equations

The first three examples in this section illustrate techniques for sketching the graph represented by a set of parametric equations. You will now investigate the reverse problem. How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is demonstrated further in the next example, which finds two different parametric representations for a given graph.

EXAMPLE 4 Finding Parametric Equations for a Given Graph

Find a set of parametric equations that represents the graph of $y = 1 - x^2$, using each of the following parameters.

a. $t = x$ **b.** The slope $m = \dfrac{dy}{dx}$ at the point (x, y)

Solution

a. Letting $x = t$ produces the parametric equations

$$x = t \quad \text{and} \quad y = 1 - x^2 = 1 - t^2.$$

b. To write x and y in terms of the parameter m, you can proceed as follows.

$$m = \frac{dy}{dx}$$

$$m = -2x \qquad \text{Differentiate } y = 1 - x^2.$$

$$x = -\frac{m}{2} \qquad \text{Solve for } x.$$

This produces a parametric equation for x. To obtain a parametric equation for y, substitute $-m/2$ for x in the original equation.

$$y = 1 - x^2 \qquad \text{Write original rectangular equation.}$$

$$y = 1 - \left(-\frac{m}{2}\right)^2 \qquad \text{Substitute } -m/2 \text{ for } x.$$

$$y = 1 - \frac{m^2}{4} \qquad \text{Simplify.}$$

So, the parametric equations are

$$x = -\frac{m}{2} \quad \text{and} \quad y = 1 - \frac{m^2}{4}.$$

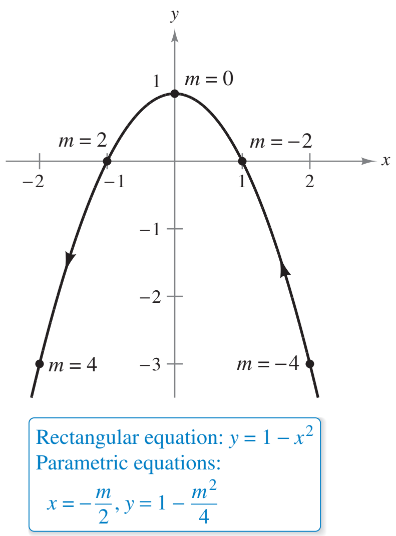


Figure 10.24

In Figure 10.24, note that the resulting curve has a right-to-left orientation as determined by the direction of increasing values of slope m. For part (a), the curve would have the opposite orientation.

FOR FURTHER INFORMATION To read about other methods for finding parametric equations, see the article "Finding Rational Parametric Curves of Relative Degree One or Two" by Dave Boyles in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

TECHNOLOGY To be efficient at using a graphing utility, it is important that you develop skill in representing a graph by a set of parametric equations. The reason for this is that many graphing utilities have only three graphing modes—(1) functions, (2) parametric equations, and (3) polar equations. Most graphing utilities are not programmed to graph a general equation. For instance, suppose you want to graph the hyperbola $x^2 - y^2 = 1$. To graph the hyperbola in *function* mode, you need two equations

$$y = \sqrt{x^2 - 1} \quad \text{and} \quad y = -\sqrt{x^2 - 1}.$$

In *parametric* mode, you can represent the graph by $x = \sec t$ and $y = \tan t$.

CYCLOIDS

Galileo first called attention to the cycloid, once recommending that it be used for the arches of bridges. Pascal once spent 8 days attempting to solve many of the problems of cycloids, such as finding the area under one arch and finding the volume of the solid of revolution formed by revolving the curve about a line. The cycloid has so many interesting properties and has caused so many quarrels among mathematicians that it has been called "the Helen of geometry" and "the apple of discord."

FOR FURTHER INFORMATION For more information on cycloids, see the article "The Geometry of Rolling Curves" by John Bloom and Lee Whitt in *The American Mathematical Monthly.* To view this article, go to *MathArticles.com.*

EXAMPLE 5 Parametric Equations for a Cycloid

Determine the curve traced by a point P on the circumference of a circle of radius a rolling along a straight line in a plane. Such a curve is called a **cycloid.**

Solution Let the parameter θ be the measure of the circle's rotation, and let the point $P = (x, y)$ begin at the origin. When $\theta = 0$, P is at the origin. When $\theta = \pi$, P is at a maximum point $(\pi a, 2a)$. When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From Figure 10.25, you can see that $\angle APC = 180° - \theta$. So,

$$\sin\theta = \sin(180° - \theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a}$$

$$\cos\theta = -\cos(180° - \theta) = -\cos(\angle APC) = \frac{AP}{-a}$$

which implies that $AP = -a\cos\theta$ and $BD = a\sin\theta$.

Because the circle rolls along the x-axis, you know that $OD = \overparen{PD} = a\theta$. Furthermore, because $BA = DC = a$, you have

$$x = OD - BD = a\theta - a\sin\theta$$
$$y = BA + AP = a - a\cos\theta.$$

So, the parametric equations are

$$x = a(\theta - \sin\theta) \quad\text{and}\quad y = a(1 - \cos\theta).$$

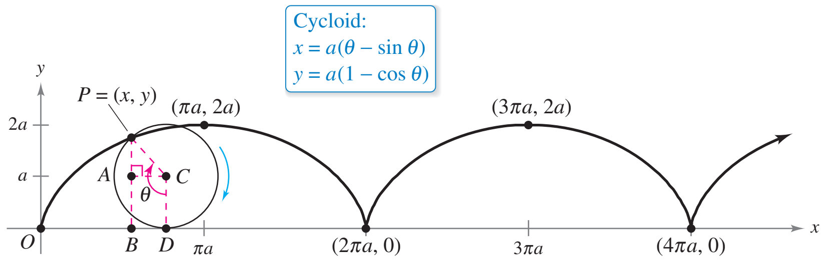


Figure 10.25

TECHNOLOGY Some graphing utilities allow you to simulate the motion of an object that is moving in the plane or in space. If you have access to such a utility, use it to trace out the path of the cycloid shown in Figure 10.25.

The cycloid in Figure 10.25 has sharp corners at the values $x = 2n\pi a$. Notice that the derivatives $x'(\theta)$ and $y'(\theta)$ are both zero at the points for which $\theta = 2n\pi$.

$$x(\theta) = a(\theta - \sin\theta) \qquad y(\theta) = a(1 - \cos\theta)$$
$$x'(\theta) = a - a\cos\theta \qquad y'(\theta) = a\sin\theta$$
$$x'(2n\pi) = 0 \qquad y'(2n\pi) = 0$$

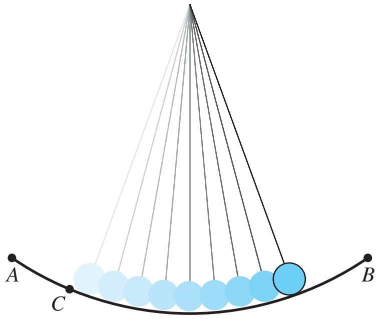
Between these points, the cycloid is called **smooth.**

Definition of a Smooth Curve

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** when f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** when it is smooth on each subinterval of some partition of I.

The Tautochrone and Brachistochrone Problems

The curve described in Example 5 is related to one of the most famous pairs of problems in the history of calculus. The first problem (called the **tautochrone problem**) began with Galileo's discovery that the time required to complete a full swing of a pendulum is *approximately* the same whether it makes a large movement at high speed or a small movement at lower speed (see Figure 10.26). Late in his life, Galileo realized that he could use this principle to construct a clock. However, he was not able to conquer the mechanics of actual construction. Christian Huygens (1629–1695) was the first to design and construct a working model. In his work with pendulums, Huygens realized that a pendulum does not take exactly the same time to complete swings of varying lengths. (This doesn't affect a pendulum clock, because the length of the circular arc is kept constant by giving the pendulum a slight boost each time it passes its lowest point.) But, in studying the problem, Huygens discovered that a ball rolling back and forth on an inverted cycloid does complete each cycle in exactly the same time.



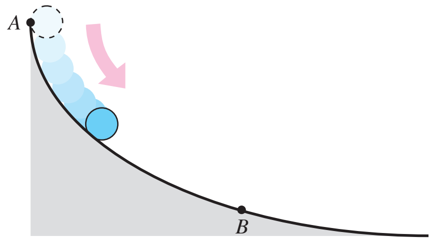
The time required to complete a full swing of the pendulum when starting from point C is only approximately the same as the time required when starting from point A.
Figure 10.26

The second problem, which was posed by John Bernoulli in 1696, is called the **brachistochrone problem**—in Greek, *brachys* means short and *chronos* means time. The problem was to determine the path down which a particle (such as a ball) will slide from point A to point B in the *shortest time*. Several mathematicians took up the challenge, and the following year the problem was solved by Newton, Leibniz, L'Hôpital, John Bernoulli, and James Bernoulli. As it turns out, the solution is not a straight line from A to B, but an inverted cycloid passing through the points A and B, as shown in Figure 10.27.

JAMES BERNOULLI (1654–1705)

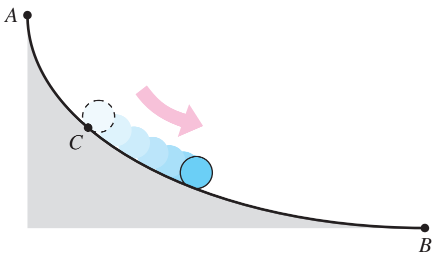
James Bernoulli, also called Jacques, was the older brother of John. He was one of several accomplished mathematicians of the Swiss Bernoulli family. James's mathematical accomplishments have given him a prominent place in the early development of calculus.

See LarsonCalculus.com to read more of this biography.



An inverted cycloid is the path down which a ball will roll in the shortest time.
Figure 10.27

The amazing part of the solution to the brachistochrone problem is that a particle starting at rest at *any* point C of the cycloid between A and B will take exactly the same time to reach B, as shown in Figure 10.28.



A ball starting at point C takes the same time to reach point B as one that starts at point A.
Figure 10.28

FOR FURTHER INFORMATION To see a proof of the famous brachistochrone problem, see the article "A New Minimization Proof for the Brachistochrone" by Gary Lawlor in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com.*

INTERFOTO/Alamy

10.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using Parametric Equations **In Exercises 1–18, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.**

1. $x = 2t - 3, \quad y = 3t + 1$
2. $x = 5 - 4t, \quad y = 2 + 5t$
3. $x = t + 1, \quad y = t^2$
4. $x = 2t^2, \quad y = t^4 + 1$
5. $x = t^3, \quad y = \dfrac{t^2}{2}$
6. $x = t^2 + t, \quad y = t^2 - t$
7. $x = \sqrt{t}, \quad y = t - 5$
8. $x = \sqrt[4]{t}, \quad y = 8 - t$
9. $x = t - 3, \quad y = \dfrac{t}{t - 3}$
10. $x = 1 + \dfrac{1}{t}, \quad y = t - 1$
11. $x = 2t, \quad y = |t - 2|$
12. $x = |t - 1|, \quad y = t + 2$
13. $x = e^t, \quad y = e^{3t} + 1$
14. $x = e^{-t}, \quad y = e^{2t} - 1$
15. $x = \sec\theta, \quad y = \cos\theta, \quad 0 \le \theta < \pi/2, \quad \pi/2 < \theta \le \pi$
16. $x = \tan^2\theta, \quad y = \sec^2\theta$
17. $x = 8\cos\theta, \quad y = 8\sin\theta$
18. $x = 3\cos\theta, \quad y = 7\sin\theta$

Using Parametric Equations **In Exercises 19–30, use a graphing utility to graph the curve represented by the parametric equations (indicate the orientation of the curve). Eliminate the parameter and write the corresponding rectangular equation.**

19. $x = 6\sin 2\theta$, $y = 4\cos 2\theta$
20. $x = \cos\theta$, $y = 2\sin 2\theta$
21. $x = 4 + 2\cos\theta$, $y = -1 + \sin\theta$
22. $x = -2 + 3\cos\theta$, $y = -5 + 3\sin\theta$
23. $x = -3 + 4\cos\theta$, $y = 2 + 5\sin\theta$
24. $x = \sec\theta$, $y = \tan\theta$
25. $x = 4\sec\theta$, $y = 3\tan\theta$
26. $x = \cos^3\theta$, $y = \sin^3\theta$
27. $x = t^3, \quad y = 3\ln t$
28. $x = \ln 2t, \quad y = t^2$
29. $x = e^{-t}, \quad y = e^{3t}$
30. $x = e^{2t}, \quad y = e^t$

Comparing Plane Curves **In Exercises 31–34, determine any differences between the curves of the parametric equations. Are the graphs the same? Are the orientations the same? Are the curves smooth? Explain.**

31. (a) $x = t$, $y = 2t + 1$ (b) $x = \cos\theta$, $y = 2\cos\theta + 1$
(c) $x = e^{-t}$, $y = 2e^{-t} + 1$ (d) $x = e^t$, $y = 2e^t + 1$

32. (a) $x = 2\cos\theta$, $y = 2\sin\theta$ (b) $x = \sqrt{4t^2 - 1}/|t|$, $y = 1/t$
(c) $x = \sqrt{t}$, $y = \sqrt{4 - t}$ (d) $x = -\sqrt{4 - e^{2t}}$, $y = e^t$

33. (a) $x = \cos\theta$, $y = 2\sin^2\theta$, $0 < \theta < \pi$ (b) $x = \cos(-\theta)$, $y = 2\sin^2(-\theta)$, $0 < \theta < \pi$

34. (a) $x = t + 1, y = t^3$ (b) $x = -t + 1, y = (-t)^3$

35. **Conjecture**
(a) Use a graphing utility to graph the curves represented by the two sets of parametric equations.
$x = 4\cos t$, $y = 3\sin t$ $\qquad$ $x = 4\cos(-t)$, $y = 3\sin(-t)$
(b) Describe the change in the graph when the sign of the parameter is changed.
(c) Make a conjecture about the change in the graph of parametric equations when the sign of the parameter is changed.
(d) Test your conjecture with another set of parametric equations.

36. **Writing** Review Exercises 31–34 and write a short paragraph describing how the graphs of curves represented by different sets of parametric equations can differ even though eliminating the parameter from each yields the same rectangular equation.

Eliminating a Parameter **In Exercises 37–40, eliminate the parameter and obtain the standard form of the rectangular equation.**

37. Line through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1)$
38. Circle: $x = h + r\cos\theta, \quad y = k + r\sin\theta$
39. Ellipse: $x = h + a\cos\theta, \quad y = k + b\sin\theta$
40. Hyperbola: $x = h + a\sec\theta, \quad y = k + b\tan\theta$

Writing a Set of Parametric Equations **In Exercises 41–48, use the results of Exercises 37–40 to find a set of parametric equations for the line or conic.**

41. Line: passes through $(0, 0)$ and $(4, -7)$
42. Line: passes through $(1, 4)$ and $(5, -2)$
43. Circle: center: $(3, 1)$; radius: 2
44. Circle: center: $(-6, 2)$; radius: 4
45. Ellipse: vertices: $(\pm 10, 0)$; foci: $(\pm 8, 0)$
46. Ellipse: vertices: $(4, 7), (4, -3)$; foci: $(4, 5), (4, -1)$
47. Hyperbola: vertices: $(\pm 4, 0)$; foci: $(\pm 5, 0)$
48. Hyperbola: vertices: $(0, \pm 1)$; foci: $(0, \pm 2)$

Finding Parametric Equations **In Exercises 49–52, find two different sets of parametric equations for the rectangular equation.**

49. $y = 6x - 5$

50. $y = 4/(x - 1)$

51. $y = x^3$

52. $y = x^2$

Finding Parametric Equations **In Exercises 53–56, find a set of parametric equations for the rectangular equation that satisfies the given condition.**

53. $y = 2x - 5$, $t = 0$ at the point $(3, 1)$

54. $y = 4x + 1$, $t = -1$ at the point $(-2, -7)$

55. $y = x^2$, $t = 4$ at the point $(4, 16)$

56. $y = 4 - x^2$, $t = 1$ at the point $(1, 3)$

Graphing a Plane Curve **In Exercises 57–64, use a graphing utility to graph the curve represented by the parametric equations. Indicate the direction of the curve. Identify any points at which the curve is not smooth.**

57. Cycloid: $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$

58. Cycloid: $x = \theta + \sin\theta$, $y = 1 - \cos\theta$

59. Prolate cycloid: $x = \theta - \frac{3}{2}\sin\theta$, $y = 1 - \frac{3}{2}\cos\theta$

60. Prolate cycloid: $x = 2\theta - 4\sin\theta$, $y = 2 - 4\cos\theta$

61. Hypocycloid: $x = 3\cos^3\theta$, $y = 3\sin^3\theta$

62. Curtate cycloid: $x = 2\theta - \sin\theta$, $y = 2 - \cos\theta$

63. Witch of Agnesi: $x = 2\cot\theta$, $y = 2\sin^2\theta$

64. Folium of Descartes: $x = 3t/(1 + t^3)$, $y = 3t^2/(1 + t^3)$

WRITING ABOUT CONCEPTS

65. **Plane Curve** State the definition of a plane curve given by parametric equations.

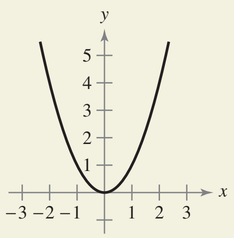
66. **Plane Curve** Explain the process of sketching a plane curve given by parametric equations. What is meant by the orientation of the curve?

67. **Smooth Curve** State the definition of a smooth curve.

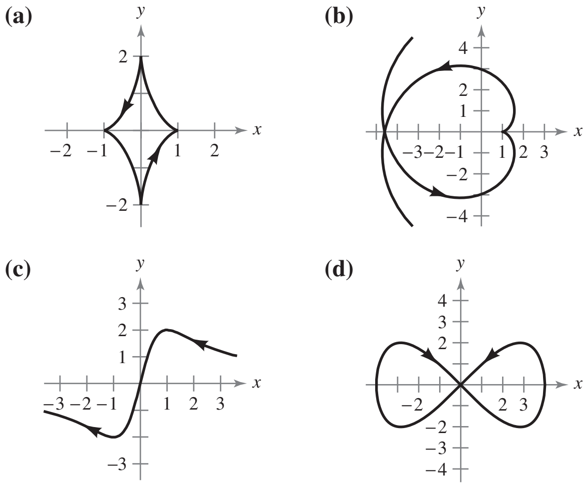
68. **HOW DO YOU SEE IT?** Which set of parametric equations is shown in the graph below? Explain your reasoning.

(a) $x = t$, $y = t^2$

(b) $x = t^2$, $y = t$



Matching **In Exercises 69–72, match each set of parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), and (d).] Explain your reasoning.**



69. Lissajous curve: $x = 4\cos\theta$, $y = 2\sin 2\theta$

70. Evolute of ellipse: $x = \cos^3\theta$, $y = 2\sin^3\theta$

71. Involute of circle: $x = \cos\theta + \theta\sin\theta$, $y = \sin\theta - \theta\cos\theta$

72. Serpentine curve: $x = \cot\theta$, $y = 4\sin\theta\cos\theta$

73. **Curtate Cycloid** A wheel of radius a rolls along a line without slipping. The curve traced by a point P that is b units from the center $(b < a)$ is called a **curtate cycloid** (see figure). Use the angle θ to find a set of parametric equations for this curve.

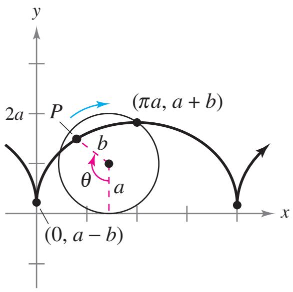


Figure for 73

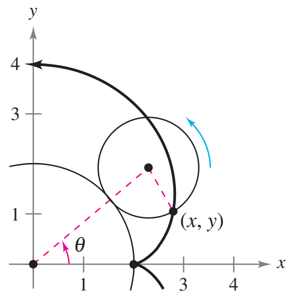


Figure for 74

74. **Epicycloid** A circle of radius 1 rolls around the outside of a circle of radius 2 without slipping. The curve traced by a point on the circumference of the smaller circle is called an epicycloid (see figure). Use the angle θ to find a set of parametric equations for this curve.

True or False? **In Exercises 75–77, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. The graph of the parametric equations $x = t^2$ and $y = t^2$ is the line $y = x$.

76. If y is a function of t and x is a function of t, then y is a function of x.

77. The curve represented by the parametric equations $x = t$ and $y = \cos t$ can be written as an equation of the form $y = f(x)$.

78. **Translation of a Plane Curve** Consider the parametric equations $x = 8 \cos t$ and $y = 8 \sin t$.

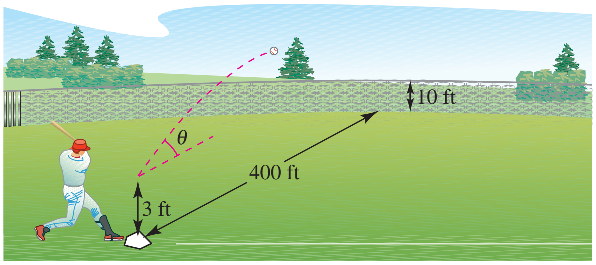
(a) Describe the curve represented by the parametric equations.

(b) How does the curve represented by the parametric equations $x = 8 \cos t + 3$ and $y = 8 \sin t + 6$ compare to the curve described in part (a)?

(c) How does the original curve change when cosine and sine are interchanged?

Projectile Motion **In Exercises 79 and 80, consider a projectile launched at a height h feet above the ground and at an angle θ with the horizontal. When the initial velocity is v_0 feet per second, the path of the projectile is modeled by the parametric equations $x = (v_0 \cos \theta)t$ and $y = h + (v_0 \sin \theta)t - 16t^2$.**

79. The center field fence in a ballpark is 10 feet high and 400 feet from home plate. The ball is hit 3 feet above the ground. It leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour (see figure).



(a) Write a set of parametric equations for the path of the ball.

(b) Use a graphing utility to graph the path of the ball when $\theta = 15°$. Is the hit a home run?

(c) Use a graphing utility to graph the path of the ball when $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle at which the ball must leave the bat in order for the hit to be a home run.

80. A rectangular equation for the path of a projectile is $y = 5 + x - 0.005x^2$.

(a) Eliminate the parameter t from the position function for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16 \sec^2 \theta}{v_0^2} x^2 + (\tan \theta)x + h.$$

(b) Use the result of part (a) to find h, v_0, and θ. Find the parametric equations of the path.

(c) Use a graphing utility to graph the rectangular equation for the path of the projectile. Confirm your answer in part (b) by sketching the curve represented by the parametric equations.

(d) Use a graphing utility to approximate the maximum height of the projectile and its range.

SECTION PROJECT

Cycloids

In Greek, the word *cycloid* means *wheel*, the word *hypocycloid* means *under the wheel,* and the word *epicycloid* means *upon the wheel.* Match the hypocycloid or epicycloid with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

Hypocycloid, H(A, B)

The path traced by a fixed point on a circle of radius B as it rolls around the *inside* of a circle of radius A

$$x = (A - B) \cos t + B \cos\left(\frac{A - B}{B}\right)t$$

$$y = (A - B) \sin t - B \sin\left(\frac{A - B}{B}\right)t$$

Epicycloid, E(A, B)

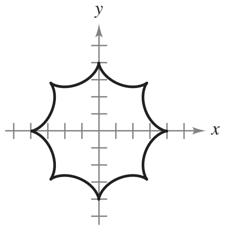
The path traced by a fixed point on a circle of radius B as it rolls around the *outside* of a circle of radius A

$$x = (A + B) \cos t - B \cos\left(\frac{A + B}{B}\right)t$$

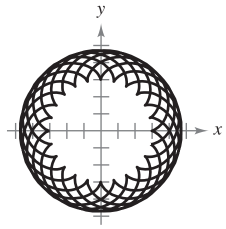
$$y = (A + B) \sin t - B \sin\left(\frac{A + B}{B}\right)t$$

I. $H(8, 3)$	II. $E(8, 3)$	III. $H(8, 7)$
IV. $E(24, 3)$	V. $H(24, 7)$	VI. $E(24, 7)$

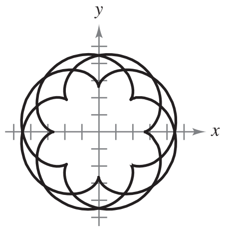
(a)



(b)



(c)



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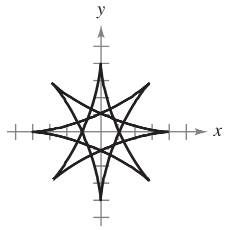
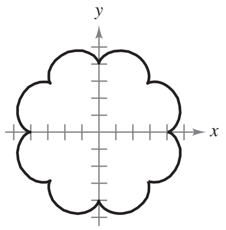
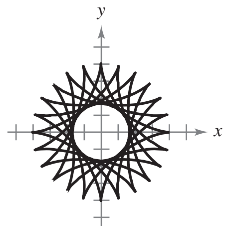


(e)



(f)



Exercises based on "Mathematical Discovery via Computer Graphics: Hypocycloids and Epicycloids" by Florence S. Gordon and Sheldon P. Gordon, *College Mathematics Journal,* November 1984, p. 441. Used by permission of the authors.